

On three open problems related to quasi relative interior

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Abstract

We give answers to two questions formulated by Borwein and Goebel in 2003 and to a conjecture formulated by Grad and Pop in 2014 related to calculus rules for quasi (relative) interior.

Key words: quasi interior, quasi relative interior, open problem.

The notion of quasi relative interior, introduced by Borwein and Lewis in 1992, became very familiar in the last ten years, being used mostly for getting necessary optimality conditions in (scalar or vector) convex programming. Unfortunately the calculus rules for the quasi relative interior is much poorer than that for other types of interiority notions. In this short note we give answers to two questions formulated by Goebel and Borwein in [1] and to a conjecture of Grad and Pop from [5] related to calculus for quasi (relative) interior.

Throughout the paper, if not specified otherwise, X is a separated locally convex space and X^* is its topological dual. For $x \in X$ and $x^* \in X^*$ we set $\langle x, x^* \rangle := x^*(x)$. Having $A \subset X$ we use the notations $\text{cl } A$, $\text{icr } A$, $\text{aff } A$, $\text{cone } A$, and $\text{lin } A$ for the closure, the intrinsic core, the affine hull, the conic hull, and the linear hull of A , respectively. Moreover, by $\overline{\text{aff}} A$, $\overline{\text{cone}} A$, and $\overline{\text{lin}} A$ we denote the closure of the sets $\text{aff } A$, $\text{cone } A$, and $\text{lin } A$, respectively. We also use $\text{lin}_0 A$ for the linear space parallel with the affine hull of A , that is, $\text{lin}_0 A := \text{aff } A - a = \text{lin}(A - a) = \text{lin}(A - A)$ for some (every) $a \in A$, and $\overline{\text{lin}}_0 A := \text{cl}(\text{lin}_0 A)$. Clearly $\overline{\text{aff}} A = \overline{\text{aff}}(\text{cl } A)$ and $\overline{\text{lin}}_0 A = \overline{\text{lin}}_0(\text{cl } A)$. For $A, B \subset X$, $a \in X$, $\Gamma \subset \mathbb{R}$ and $\gamma \in \mathbb{R}$ we set

$$\begin{aligned} A + B &:= \{a + b \mid a \in A, b \in B\}, \quad a + A := \{a\} + A, \\ \gamma A &:= \{\gamma a \mid a \in A\} \text{ for } \gamma \neq 0, \quad 0A := \{0\}, \quad \Gamma A := \cup_{\alpha \in \Gamma} \alpha A, \quad \Gamma a := \Gamma\{a\}. \end{aligned}$$

Therefore, $A + \emptyset = \emptyset + A = \emptyset$ and $0 \in \text{cone } A$ for any set $A \subset X$.

First we recall the notions of quasi interior and quasi relative interior for convex sets and some properties of these notions.

Let $C \subset X$ be a convex set; the *quasi interior* of C (see [1, p. 2544]) is the set

$$\text{qi } C := \{x \in C \mid \overline{\text{cone}}(C - x) = X\},$$

and the *quasi relative interior* of C (see [2, Def. 2.3]) is the set

$$\text{qri } C := \{x \in C \mid \overline{\text{cone}}(C - x) \text{ is a linear space}\};$$

hence $\text{qi } \emptyset = \text{qri } \emptyset = \emptyset$. It follows (see [6, Prop. 1.2.7 (1.4)]) that

$$\text{qri } C = \{x \in C \mid \overline{\text{cone}}(C - x) = \overline{\text{lin}}_0 C\};$$

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therefore, because $\overline{\text{aff}} C = X$ iff $\overline{\text{lin}}_0 C = X$,

$$\text{qi } C = \begin{cases} \text{qri } C & \text{if } \overline{\text{aff}} C = X, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence (see also [3, Lem. 6]),

$$\text{qi } C \neq \emptyset \Rightarrow 0 \in \text{qi}(C - C) \Leftrightarrow \overline{\text{aff}} C = X \Leftrightarrow \overline{\text{lin}}_0 C = X \Rightarrow \text{qi } C = \text{qri } C. \quad (1)$$

Observe (see [6, Prop. 1.2.8 (ii)]) that for $C \neq \emptyset$ we have:

$$[x \in X, \overline{\text{cone}}(C - x) \text{ is a linear space}] \Rightarrow x \in \text{cl } C. \quad (2)$$

Because $\overline{\text{cone}}(C - x) = \overline{\text{cone}}(\text{cl } C - x)$ for every $x \in X$, from the definition of $\text{qri } C$ and (2), for $C \neq \emptyset$, we get

$$\text{qri}(\text{cl } C) = \{x \in X \mid \overline{\text{cone}}(C - x) \text{ is a linear space}\} = \{x \in X \mid \overline{\text{cone}}(C - x) = \overline{\text{lin}}_0 C\},$$

whence

$$\text{qi } C = C \cap \text{qi}(\text{cl } C), \quad \text{qri } C = C \cap \text{qri}(\text{cl } C); \quad (3)$$

hence, if $A, B \subset X$ are convex sets, then

$$A \subset B \subset \text{cl } A \Rightarrow \text{qri } A \subset \text{qri } B \subset \text{qri}(\text{cl } A).$$

The facts that $(1 - \lambda)C + \lambda \text{qri } C \subset \text{qri } C$ for $\lambda \in (0, 1)$ and $\text{qri}(x + C) = x + \text{qri } C$ for $x \in X$ are well known (see, e.g., [2]), and so $\text{qri } C$ is convex and, $\text{cl}(\text{qri } C) = \text{cl } C$ if $\text{qri } C \neq \emptyset$; therefore, $\overline{\text{lin}}_0(\text{qri } C) = \overline{\text{lin}}_0 C$ provided $\text{qri } C \neq \emptyset$. It follows (see [4, Prop. 2.5 (vii)]) that

$$\text{qi}(\text{qi } C) = \text{qi } C, \quad \text{qri}(\text{qri } C) = \text{qri } C. \quad (4)$$

Indeed, assume that $\text{qri } C \neq \emptyset$. By (3),

$$\text{qri}(\text{qri } C) = \text{qri } C \cap \text{qri}(\text{cl}(\text{qri } C)) = \text{qri } C \cap \text{qri}(\text{cl } C) = \text{qri } C.$$

Relation (5) in the next result is stated in [5, Prop. 5] for D a pointed convex cone with $\text{qi } D \neq \emptyset$.

Proposition 1 *Let $C, D \subset X$ be convex sets; then*

$$C + \text{qi } D = \text{qi}(C + \text{qi } D) \subset \text{qi}(C + D), \quad (5)$$

$$\text{qri } C + \text{qri } D = \text{qri}(\text{qri } C + \text{qri } D) \subset \text{qri}(C + D) \quad (6)$$

Proof. Relation (5) is obvious if $\text{qi } D = \emptyset$. Assume that $\text{qi } D \neq \emptyset$. By (4), for $x \in C$ we have that $\text{qi}(x + \text{qi } D) = x + \text{qi}(\text{qi } D) = x + \text{qi } D \subset \text{qi}(C + \text{qi } D)$. Hence

$$\text{qi}(C + \text{qi } D) \subset C + \text{qi } D = \cup_{x \in C} (x + \text{qi } D) \subset \text{qi}(C + \text{qi } D) \subset \text{qi}(C + D),$$

and so (5) holds.

Take $x \in \text{qri } C$ and $y \in \text{qri } D$. Since $\text{cl}(\text{qri } C) = \text{cl } C$, we have $\overline{\text{lin}}_0(\text{qri } C) = \overline{\text{lin}}_0 C$ and $\overline{\text{cone}}(\text{qri } C - x) = \overline{\text{lin}}_0 C$; similarly for D and y . Hence

$$\begin{aligned} \overline{\text{cone}}(\text{qri } C + \text{qri } D) - (x + y) &= \text{cl}[\text{cone}(\text{qri } C - x) + \text{cone}(\text{qri } D - y)] \\ &= \text{cl}[\overline{\text{cone}}(\text{qri } C - x) + \overline{\text{cone}}(\text{qri } D - y)] \\ &= \text{cl}(\overline{\text{lin}}_0 C + \overline{\text{lin}}_0 D) = \overline{\text{lin}}_0(C + D) \\ &= \overline{\text{lin}}_0(\text{qri } C + \text{qri } D). \end{aligned}$$

It follows that $x + y \in \text{qri}(\text{qri } C + \text{qri } D)$. Hence $\text{qri } C + \text{qri } D = \text{qri}(\text{qri } C + \text{qri } D)$. The inclusion $\text{qri } C + \text{qri } D \subset \text{qri}(C + D)$ is well known (see [1, Lem. 3.6 (b)]). Hence (6) holds. \square

Taking into account (1), from (5) one obtains [4, Lem. 2.6] and [3, Lem. 6]).

Borwein and Goebel, in [1, p. 2548], say “*Can $\text{qri } C + \text{qri } D$ be a proper subset of $\text{qri}(C + D)$? (Almost certainly such sets do exist.)*”, while Grad and Pop, in [5, p. 26], say: “*we conjecture that in general when $A, B \subseteq V$ are convex sets with $\text{qi } B \neq \emptyset$, it holds $A + \text{qi } B = \text{qi}(A + B)$* ”. The next example answers to both problems mentioned above.

Example 2 Take $X := \ell_2 := \{(x_n)_{n \geq 1} \in \mathbb{R} \mid \sum_{n \geq 1} x_n^2 < \infty\}$ endowed with its usual norm, $\bar{x} := (n^{-1})_{n \geq 1} \in \ell_2$, $C := [0, 1]\bar{x} \subset \ell_2$ and $D := \ell_1^+ := \{(x_n)_{n \geq 1} \in \mathbb{R}_+ \mid \sum_{n \geq 1} x_n < \infty\} \subset \ell_2$. Clearly C and D are convex sets, $\text{qri } C = \text{icr } C = (0, 1)\bar{x}$, $\text{qri } D = \text{qi } D = \{(x_n)_{n \geq 1} \in \ell_1 \mid x_n > 0 \ \forall n \geq 1\}$ and $\bar{x} \in \text{qi}(C + D) = \text{qri}(C + D)$, but $\bar{x} \notin C + \text{qi } D \supset \text{qri } C + \text{qri } D$.

Proof. First observe that $\text{aff } D = \text{lin}_0 D = D - D = \ell_1$ and $\text{cl } D = \ell_2^+$. Therefore, $\overline{\text{lin}}_0 D = \ell_2$, and so, using (3),

$$\text{qri } D = \text{qi } D = D \cap \text{qi } \ell_2^+ = \{x \in \ell_1 \mid x_n > 0 \ \forall n \geq 1\}. \quad (7)$$

Clearly, $\ell_1^+ \subset C + D \subset \ell_2^+ + \ell_2^+ = \ell_2^+$, whence $\text{cl}(C + D) = \ell_2^+$. Since $\bar{x} \in (C + D) \cap \text{qi } \ell_2^+$, we obtain that $\bar{x} \in \text{qi}(C + D)$ using (3). Assume that $\bar{x} \in C + \text{qi } D$. Then $\bar{x} \in t\bar{x} + \text{qi } D \subset t\bar{x} + \ell_1$ with $t \in [0, 1]$, whence $(1 - t)\bar{x} \in \ell_1$. Because $\bar{x} \notin \ell_1$, we have that $t = 1$, and so $0 \in \text{qi } D$. This is a contradiction by (7). The conclusion follows. \square

It is worth observing that for $x_0 \in C$ we have that

$$x_0 \notin \text{qi } C \iff \exists x^* \in X^* \setminus \{0\} : \inf x^*(C) = \langle x_0, x^* \rangle,$$

that is x_0 is a support point of C , and

$$x_0 \notin \text{qri } C \iff \exists x^* \in X^* : \sup x^*(C) > \inf x^*(C) = \langle x_0, x^* \rangle \quad (8)$$

(see also [1, Lem. 2.7]); in particular,

$$x_0 \in C \setminus \text{qri } C \implies \exists x^* \in X^* \setminus \{0\} : \inf x^*(C) = \langle x_0, x^* \rangle.$$

Note that in the above implications we do not assume that $\text{qri } C \neq \emptyset$.

Proposition 3 Let $C, D \subset X$ be nonempty convex sets.

- (i) If $C \cap \text{qri } D \neq \emptyset$ then $\text{qri}(C \cap D) \subset C \cap \text{qri } D$.
- (ii) If $\text{qri } C \cap \text{qri } D \neq \emptyset$ then $\text{qri}(C \cap D) \subset \text{qri } C \cap \text{qri } D$.

Proof. (i) Fix $x_0 \in C \cap \text{qri } D$ ($\subset C \cap D$). Consider $x \in \text{qri}(C \cap D)$ ($\subset C \cap D$). Suppose that $x \notin \text{qri } D$. By (8), there exists $x^* \in X^* \setminus \{0\}$ such that

$$\langle x, x^* \rangle = \inf x^*(D) < \sup x^*(D). \quad (9)$$

Because $x \in C \cap D$, it follows that $\langle x, x^* \rangle = \inf x^*(C \cap D)$. Since $x \in \text{qri}(C \cap D)$, using again (8), we have that $\langle x, x^* \rangle = \langle y, x^* \rangle$ for every $y \in C \cap D$, whence $\langle x, x^* \rangle = \langle x_0, x^* \rangle$. From (9) we get $\langle x_0, x^* \rangle = \inf x^*(D) < \sup x^*(D)$ which implies, by (8), that $0 \notin \text{qri } D$. This contradiction proves that $\text{qri}(C \cap D) \subset C \cap \text{qri } D$.

(ii) Assume that $\text{qri } C \cap \text{qri } D \neq \emptyset$. Then $C \cap \text{qri } D \neq \emptyset$ and $\text{qri } C \cap D \neq \emptyset$. By (i) we get

$$\text{qri}(C \cap D) \subset (C \cap \text{qri } D) \cap (\text{qri } C \cap D) = \text{qri } C \cap \text{qri } D.$$

The proof is complete. \square

Borwein and Goebel, in [1, p. 2548], also say “*Can $\text{qri}(C \cap D) \subset \text{qri } C \cap \text{qri } D$ fail when $\text{qri } C \cap \text{qri } D \neq \emptyset$?*” Proposition 3 (ii) shows that the answer to this question is negative.

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